Well-posedness in H^1 for the (generalized) Benjamin-Ono equation on the circle

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Abstract. We prove the local well posedness of the Benjamin-Ono equation and the generalized Benjamin-Ono equation in $H^1(\mathbb{T})$. This leads to a global well-posedness result in $H^1(\mathbb{T})$ for the Benjamin-Ono equation.

1 Introduction, main results and notations

1.1 Introduction and main results

In this paper we study the $H^1(\mathbb{T})$ local well-posedness problem for the Benjamin-Ono equation and the generalized Benjamin-Ono equation

$$\text{(GBO)}\qquad \left\{\begin{array}{l} \partial_t u + \mathcal{H}\partial_{xx} u = u^k \partial_x u \ , \ (t,x) \in \mathbb{R}\times\mathbb{T} \ , \\ u(0,x) = u_0(x) \ . \end{array}\right.$$

Here $\mathbb{T} = \mathbb{R}/(2\pi\mathbb{Z})$, $k \geq 1$ is an integer and $\mathcal{H}$ denotes the Hilbert transform defined for $2\pi\lambda$-periodic functions by

$$\left\{\begin{array}{l} \widehat{\mathcal{H}}(f)(q) = -i\,\mathrm{sgn}(q)\hat{f}(q)\ , \ q \in \lambda^{-1}\mathbb{Z}^* \ , \\ \widehat{\mathcal{H}}(f)(0) = 0 \ . \end{array}\right.$$

When $k = 1$, (GBO) is the well known Benjamin-Ono equation. This equation has been derived as a model for the propagation of long internal gravity waves in deep and stratified fluids [3], and, at least when the spatial domain is the whole real line, has been studied in a large amount of works in

the last decades. The Benjamin-Ono equation is a totally integrable system [8], and possesses among others the three following invariant quantities :

$$I(u)=\int u(t,x)\,dx\,,\quad M(u)=\int u^2(t,x)\,dx\,,$$

and

$$F(u)=\int u_x^2(t,x)-\frac{3}{4}\,u^2(t,x)\mathcal{H}u_x(t,x)-\frac{1}{8}\,u^4(t,x)\,dx\quad.$$

Using Sobolev embedding theorems together with standard interpolation inequalities, it is straightforward to check that these conservation laws lead to the following $H^1(\mathbb{R})$ a priori estimate for regular solutions of the (BO) equation,

$$\forall t\geq 0\,,\ \|u(t)\|_{H^1}\leq C\,\|u_0\|_{H^1}\quad. \tag{1}$$

Thus, any local well possedness result in $H^1(\mathbb{R})$ for the (BO) equation can be extend to a global one.

In our knowledge the first results concerning the well possedness of (BO) in the Sobolev spaces $H^s(\mathbb{R})$ have been obtained in [22] where the global well posedness is proved in $H^3(\mathbb{R})$. This was improved later to a global well posedness result in $H^s(\mathbb{R})$, $s>3/2$ in [1], [11] and next in $H^{3/2}(\mathbb{R})$, see [21]. Then, by means of some dispersive estimates for the non homogeneous linear Benjamin-Ono equation, (BO) has been proved to be locally well posed in $H^s(\mathbb{R})$, $s>5/4$ in [16] and next in $H^s(\mathbb{R})$, $s>9/8$, [13]. Recently T. Tao [23] get the global well posedness in $H^1(\mathbb{R})$ by using a gauge transformation together with the well known Strichartz estimate

$$\|V(t)\varphi\|_{L^4_{t,x}}\leq C\,\|\varphi\|_{L^2}\,, \tag{2}$$

where $V(\cdot)$ denotes the free linear group associated to the linear Benjamin-Ono equation. Up to now the best result concerning this problem is due to A.D. Ionescu and C.E. Kenig who obtained very recently the global well posedness of (BO) in $L^2(\mathbb{R})$, [12].

It is worth noticing that all these recent results have been obtained by coupling compactness methods together with "smoothing" estimates for $V(\cdot)$. It is also important to notice that, for all $s\in\mathbb{R}$, the flow map is not of class C^2 from $H^s(\mathbb{R})$ to $C([0,T],H^s(\mathbb{R}))$, [20]. Actually it has recently been proved in [17] that, for all $s>0$, the flow map is not even uniformly continuous on bounded sets of $H^s(\mathbb{R})$. This is mainly due to some bad interactions between low and high frequencies in the nonlinear term uu_x as pointed out in [20] and [17] and this explains why contraction methods can not be used to solve (BO) in $H^s(\mathbb{R})$.

Concerning the periodic case, the global well posedness in $H^s(\mathbb{T})$, $s > 3/2$ is derived in [1]. It is worth noticing that the proof did not use the smoothing properties of $V(\cdot)$. Recall that in the periodic case both the dispersive estimates

$$\|V(t)\varphi\|_{L^\infty_x} \lesssim t^{-1/2}\, \|\varphi\|_{L^1}\ , \tag{3}$$

and the sharp Kato smoothing effect

$$\|D_x^{1/2} V(t)\varphi\|_{L^\infty_x L^2_t} \lesssim \|\varphi\|_{L^2} \tag{4}$$

fail. This probabely explains why there is no result concerning the periodic Cauchy problem in $H^s(\mathbb{T})$, $s \le 3/2$, for the (BO) equation. In this paper, following the work of T. Tao [23], we use a gauge transformation together with the periodic estimate (8) proved in [6] to obtain the following result (see subsection 1.2 below for the definition of the space X^1_T) :

Theorem 1.1 *For all $u_0 \in H^1(\mathbb{T})$ and all $T > 0$, there exists a unique global solution u of the Benjamin-Ono equation in*

$$X^1_T \cap C_b(\mathbb{R}, H^1(\mathbb{T}))\ .$$

Moreover, the flow-map is continuous from $H^1(\mathbb{T})$ to $C([0,T], H^1(\mathbb{T}))$ and, for all $\gamma \in \mathbb{R}$, is Lipschitz on every bounded subset of Γ_γ where

$$\Gamma_\gamma = \{ f \in H^1(\mathbb{T})\, , \ \fint f(x)\, dx = \gamma \}\, .$$

Remark 1.1 *To prove the above Lipschitz property and the uniqueness part of Theorem 1.1, we will use in a crucial way that, in sharp contrast with the non periodic case, the gauge transformation is in fact a Lipschitz map from the set of L^2 functions with zero mean value on the circle into L^∞. This also avoid to consider some frequency enveloppe considerations as done in [23].*

When $k \ge 2$, (GBO) is no more a totally integrable system and there is no conservation law at the level H^1. Nonetheless we still have the three following quantities conserved by the flow,

$$I(u) = \int u(t,x)\, dx\, , \quad M(u) = \int u^2(t,x)\, dx\, ,$$

and

$$\tilde{E}(u) = \int \Big(\frac{1}{2}\, |D_x^{1/2} u(t,x)|^2 \mp \frac{1}{(k+1)(k+2)}\, u(t,x)^{k+2} \Big)\, dx \ \ (energy)\ \ .$$

In [19], in the case of the whole real line, by means of a gauge transformation together with some linear estimates, we proved the local well posedness of (GBO) on the line in $H^s(\mathbb{R})$, $s \geq 1/2$ for $k \geq 5$, in $H^s(\mathbb{R})$, $s > 1/2$ for $k = 2, 4$ and in $H^s(\mathbb{R})$, $s \geq 3/4$ for $k = 3$ [1]. In all those cases we also obtained that, in a sharp contrast with the (BO) equation, the flow map is lipschitz on bounded set of $H^s(\mathbb{R})$ which has to be viewed as a stability result for the (GBO) equation when $k \neq 1$.

In the periodic case, using again the proofs given in [11], it is straightforward to derive the local well posedness of (GBO) in $H^s(\mathbb{T})$ when $s > 3/2$. On the contrary, up to our knowledge, there is no available result on this problem when $s \leq 3/2$. As for the (BO) equation, this is probably due to the failure of the dispersive estimate (3) and the sharp Kato smoothing effect (4). Again, using a gauge transformation and the periodic estimate (8) we prove the following result,

Theorem 1.2 *Let $k \geq 2$ be an integer. For all $u_0 \in H^1(\mathbb{T})$ there exists $T = T(\|u_0\|_{H^1}) > 0$ and a unique solution u of (GBO) in*

$$X_T^1 \cap C([0,T], H^1(\mathbb{T})) \ .$$

Moreover, the flow-map is continuous from $H^1(\mathbb{T})$ to $C([0,T], H^1(\mathbb{T}))$.

1.2 Notations

In the sequel C denotes a positive constant which may differ at each appearance. When writing $x \lesssim y$ (for x and y two nonnegative real numbers), we mean that there exists C_1 a positive constant (which does not depend of x and y) such that $x \leq C_1 y$.

We will use the space-time Lebesgues spaces $L^p_T L^r_\lambda$ of the $2\pi\lambda$-periodic function (in space) endowed with the norms

$$\|f(t,x)\|_{L^p_T L^r_\lambda} = \| \, \|f(t,\cdot)\|_{L^r([0,2\pi\lambda])} \|_{L^p([-T,+T])} \ .$$

When $p = r$, we rather use the notation $L^r_{T,\lambda} = L^r_T L^r_\lambda$.
We will also need the functional spaces X_T^0, X_T^1 and X_T^2 respectively defined trough the norms

$$\|u\|_{X_\lambda^0} = \|u\|_{L_T^\infty L_\lambda^2} + \|u\|_{L^4_{T,\lambda}} \ ,$$

$$\|u\|_{X_\lambda^1} = \|u\|_{X_\lambda^0} + \|u_x\|_{L_T^\infty L_\lambda^2} + \|u_x\|_{L^4_{T,\lambda}} \ ,$$

[1] See also [18] where optimal results are obtained for (GBO) by contraction methods in the particular context of small initial data.

and

$$\|u\|_{X^2_\lambda} = \|u\|_{X^1_\lambda} + \|u_{xx}\|_{L^\infty_T L^2_\lambda} + \|u_{xx}\|_{L^4_{T,\lambda}} \; .$$

In a standard way H^s_λ denotes the space of $2\pi\lambda$-periodic functions such that

$$\|f\|_{H^s_\lambda} = \Big(\sum_{q\in\mathbb{Z}/\lambda} (1+q^2)^s \, |C_q(f)|^2 \Big)^{1/2} < +\infty \; ,$$

where, for $q \in \lambda^{-1}\mathbb{Z}$,

$$C_q(f) = \frac{1}{2\pi\lambda} \int_0^{2\pi\lambda} f(t) e^{-iqt} \, dt \; .$$

Also, for a $2\pi\lambda$-periodic function f we respectively defined the projection operators P_+, P_-, P_k and $P_{>k}$ by

$$P_+(f) = \sum_{q\in\mathbb{Z}^*_+/\lambda} C_q(f) \, e^{iqx} \; , \; P_-(f) = \sum_{q\in\mathbb{Z}^*_-/\lambda} C_q(f) \, e^{iqx} \; ,$$

$$P_k(f) = \sum_{q\in\mathbb{Z}/\lambda, \, |q|\le k} C_q(f) \, e^{iqx} \quad \text{and} \quad P_{>k}(f) = \sum_{q\in\mathbb{Z}/\lambda, \, q>k} C_q(f) \, e^{iqx} \; .$$

2 Linear estimates

Let us first recall the following estimate established by Bourgain [6] (it was proven for the Schrödinger group but the adaptation for the Benjamin-Ono group is straightforward).

$$\|V(t)\varphi\|_{L^4_1 L^4_1} \lesssim \|\varphi\|_{L^2_1} \quad . \tag{5}$$

Of course (5) still holds for any period $\lambda \sim 1$. On the other hand, by a scaling argument, it is clear that pushing λ to $+\infty$, a factor $\lambda^{1/4}$ will appear at the right-hand side of (5). Since to solve the Cauchy problem for large initial data, we will use a rescaling argument that will make us work with a large period, the estimate (5) will not be sufficient. We will rely instead on the following improved Zygmund estimate also shown in [6] :

$$\|u\|_{L^4_1 L^4_1} \lesssim \|u\|_{X_1^{3/8,0}} \quad . \tag{6}$$

Here the $X^{s,b}_1$ are the function spaces introduced by Bourgain to solve the Cauchy problem for the periodic Schrödinger equation. Recall that

$$\|u\|_{X_1^{b,s}} = \|U(-t)u(t)\|_{H_1^{b,s}} \quad , \tag{7}$$

where $U(t)$ is the Schrödinger one parameter linear group. Again, by separating the positive and the negative frequencies of u and using (6), it is obvious to see that (6) still holds when replacing $U(\cdot)$ by $V(\cdot)$ in the definition of $X^{s,b}$(see (7)). With (6) in hand, we establish now the following lemma that gives a periodic estimate which turns out to be uniform with respect to large periods λ.

Lemma 2.1 *There exists a constant $C>0$ such that $\forall \lambda \geq 1$, $\forall \varphi \in L^2_\lambda$,*

$$\|V(t)\varphi\|_{L^4_{[0,1]}L^4_\lambda} \leq C\|\varphi\|_{L^2_\lambda} \quad . \tag{8}$$

Proof . We first take $\lambda = 1$. Let ψ be a C^∞-function such that $0 \leq \psi \leq 1$, $\psi \equiv 1$ on $[-1/4, 1/4]$ and $\operatorname{supp}\psi \subset [-1/2, 1/2]$. We set $\psi_T(\cdot) = \psi(\cdot/T)$. Of course, $\psi_T \equiv 1$ on $[-\frac{T}{4}, \frac{T}{4}]$ and $\operatorname{supp}\psi_T \subset [-\frac{T}{2}, \frac{T}{2}]$. Moreover, for $0 < T \leq 1$, ψ_T can be extended outside $[-1/2, 1/2]$ to a 1-periodic function. Therefore, by (6),

$$\begin{aligned} \|V(t)\varphi\|_{L^4_{[-\frac{T}{4},\frac{T}{4}]}L^4_1} &\lesssim \|\psi_T\, V(t)\varphi\|_{L^4_1 L^4_1} = \|V(t)(\psi_T\varphi)\|_{L^4_1 L^4_1} \\ &\lesssim \|\psi_T\varphi\|_{X_1^{3/8,0}} \\ &\lesssim \|\psi_T\|_{H_1^{3/8}}\|\varphi\|_{L^2_1} \lesssim T^{1/8}\|\varphi\|_{L^2_1} \quad . \end{aligned} \tag{9}$$

Where in the last step we use that

$$\|\psi_T\|_{H_1^{3/8}} \lesssim \|\psi_T\|_{L^2_1} + \|\psi_T\|_{\dot{H}_1^{3/8}} \lesssim T^{1/8} \quad .$$

Now for $\varphi_\lambda \in L^2_\lambda$, we define $\varphi(\cdot) = \lambda\varphi_\lambda(\lambda\cdot) \in L^2_1$. Setting $u_\lambda(t,x) = (V(t)\varphi_\lambda)(x)$ and $u(t,x) = (V(t)\varphi)(x)$, we easily check that $u(t,x) = \lambda u_\lambda(\lambda^2 t, \lambda x)$. Hence, taking $T = 4\lambda^{-2}$ in (9), we get

$$\begin{aligned} \|V(t)\varphi_\lambda\|_{L^4_{[0,1]}L^4_\lambda} &= \lambda^{-1/4}\|V(t)\varphi\|_{L^4_{[0,1/\lambda^2]}L^4_1} \\ &\lesssim \lambda^{-1/2}\|\varphi\|_{L^2_1} \\ &\lesssim \|\varphi_\lambda\|_{L^2_\lambda} \quad . \end{aligned} \tag{10}$$

This completes the proof of the lemma.

3 Proof of Theorem 1

We start by the proof of the local well-posedness result for the Benjamin-Ono equation which is much simpler than the one for (GBO). This is mainly

due to the three following facts :
• The equation satisfied by the gauge transform is simpler.
• There exists a global existence result for smooth initial data.
• The L^2-norm is surcritical for (BO).

3.1 Gauge transformation and nonlinear estimates

Let $\lambda \geq 1$ and u be a global H^∞_λ-solution of (BO) with initial data u_0. In the sequel, we assume that u_0 has zero mean value. Otherwise we do the change of unknown

$$v(t,x) = u(t, x - t \fint u) - \fint u \quad . \tag{11}$$

Since $\fint u$ is preserved by the flow, it is straightforward to see that v satisfies (BO) with $v_0 = u_0 - \fint u_0$ as initial data and that $\fint v = 0$. Hence we are reduced to the case of a zero mean-value initial data and thus to the case of zero mean-value solutions. Also, changing u to $u/2$, we can always assume that u satisfies the equation,

$$u_t + \mathcal{H}u_{xx} = 2uu_x \ .$$

We define $F = \partial_x^{-1} u$ which is the periodic, zero mean value, primitive of u and following T. Tao [23], we introduce the gauge transform

$$W = P_+(e^{-iF}) \quad ,$$

and we consider

$$w = W_x = -iP_+(e^{-iF} F_x) = -iP_+(e^{-iF} u) \ .$$

Following the calculations performed in subsection 4.1 (for $k = 1$) and noticing that

$$P_+(e^{-iF} P_-(u_{xx})) - iP_+(ue^{-iF} P_-(u_x)) = \partial_x P_+ \Big(e^{-iF} P_-(F_{xx}) \Big) \, ,$$

and that

$$\partial_x^{-1} \, \partial_x(u^2) = u^2 - P_0(u^2) \, ,$$

we then obtain that w solves the following Schrödinger equation :

$$\begin{aligned} w_t - i w_{xx} &= -2\,\partial_x P_+\Big(P_-(F_{xx})e^{-iF}\Big) + P_0(u^2)P_+(ue^{-iF}) \\ &= -2\partial_x P_+\Big(P_-(u_x)e^{-iF}\Big) + P_0(u^2)P_+(ue^{-iF}) \quad . \end{aligned} \tag{12}$$

Using the Duhamel formulation of this Schrödinger equation on w it follows from (8) and standard TT^* arguments that

$$\|w\|_{X^1_{T,\lambda}} \le \|w(0)\|_{H^1_\lambda} + \| -2\partial_x P_+\Big(P_-(u_x)e^{-iF}\Big) + P_0(u^2)P_+(ue^{-iF})\|_{L^1_T H^1_\lambda}$$

Now, from Lemma 4.1 and the estimate for $|P_0(u^2)|_{L^\infty_T}$ derived in subsection 4.2, we obtain that, for $0 < T \le 1$,

$$\begin{aligned} \|w\|_{X^1_{T,\lambda}} &\lesssim \|w(0)\|_{H^1_\lambda} + \|\partial_x P_+(P_-(u_x)W)\|_{L^1_T H^1_\lambda} \\ &+ |P_0(u^2)|_{L^\infty_T}\|P_+(u\,e^{-iF})\|_{L^1_T H^1_\lambda} \\ &\lesssim \|w(0)\|_{H^1_\lambda} + T^{1/2}\,\|u_x\|_{L^4_{T,\lambda}}\|J^1_x w\|_{L^4_{T,\lambda}} \\ &+ T\|u\|_{X^1_{T,\lambda}}(\|u\|_{X^1_{T,\lambda}} + \|u\|^2_{X^1_{T,\lambda}}) \end{aligned} \tag{13}$$

which leads to

$$\begin{aligned} \|w\|_{X^1_{T,\lambda}} &\lesssim \|u_0\|_{H^1_\lambda} + \|u_0\|^2_{H^1_\lambda} \\ &+ T^{1/2}\|u\|_{X^1_{T,\lambda}}(\|u\|_{X^1_{T,\lambda}} + \|u\|^2_{X^1_{T,\lambda}} + \|w\|_{X^1_{T,\lambda}}) \quad . \end{aligned} \tag{14}$$

On the other hand, we can rewrite u as

$$u = e^{iF}e^{-iF}u = e^{iF}P_+(e^{-iF}u) + e^{iF}P_-(e^{-iF}u) \quad , \tag{15}$$

and so,

$$\begin{aligned} P_{>1}u &= iP_{>1}\Big(e^{iF}w\Big) + P_{>1}\Big(e^{iF}P_-(e^{-iF}u)\Big) \\ &= iP_{>1}\Big(e^{iF}w\Big) + P_{>1}\Big(P_{>1}(e^{iF})P_-(e^{-iF}u)\Big) \quad . \end{aligned}$$

Hence from Lemma 4.1 (and since u is real-valued), we infer that

$$\begin{aligned} \|u\|_{X^1_{T,\lambda}} &= \|P_1 u\|_{X^1_{T,\lambda}} + 2\|P_{>1}\,u\|_{X^1_{T,\lambda}} \\ &\lesssim \|P_1 u\|_{X^1_{T,\lambda}} + \|w\|_{X^1_{T,\lambda}} + \|u\|_{X^1_{T,\lambda}}\|w\|_{X^0_{T,\lambda}} \\ &+ \|P_{>1}(e^{iF})\|_{L^\infty_{T,\lambda}}\|u\|_{X^1_{T,\lambda}} + \|u\|^2_{X^1_{T,\lambda}} \\ &\lesssim \|P_1 u\|_{X^1_{T,\lambda}} + \|w\|_{X^1_{T,\lambda}} + \|u\|_{L^\infty_{T,\lambda}}\|u\|_{X^1_{T,\lambda}} + \|u\|^2_{X^1_{T,\lambda}} \end{aligned} \tag{16}$$

where in the last step we use that, by Bernstein's inequality for $2\pi\lambda$-periodic functions,

$$\|P_{>1}(e^{iF})\|_{L^\infty_{T,\lambda}} \lesssim \|\partial_x(e^{iF})\|_{L^\infty_{T,\lambda}} = \|e^{iF}F_x\|_{L^\infty_{T,\lambda}} = \|F_x\|_{L^\infty_{T,\lambda}} = \|u\|_{L^\infty_{T,\lambda}} ,$$

and that, $\|w\|_{X^0_{T,\lambda}} \le \|u\|_{X^0_{T,\lambda}}$.

3.2 Local well-posedness for small data

We will now prove the local well-posedness result for small initial data. The result for arbitrary large data will follow from scaling arguments.

3.2.1 Existence

Let $u_0 \in H^\infty_\lambda$ be a $2\pi\lambda$-periodic zero mean-value function and let us assume that $\|u_0\|_{H^1_\lambda} \lesssim \varepsilon^2$ for some small $0 < \varepsilon \ll 1$ depending only on the implicit constant contained in the above estimates. At this stage, it is worth recalling that these implicit constants do not depend on the period λ.

Our aim is to show that the emanating solution $u \in C(\mathbb{R}; H^\infty_\lambda)$ satisfies $\|u\|_{X^1_{1,\lambda}} \lesssim \varepsilon^2$.

First, since the L^2-norm of u is constant along the trajectory, we obtain from Bernstein inequalities that

$$\|P_1 u\|_{X^1_{1,\lambda}} \lesssim \|u\|_{L^\infty_1 L^2_x} = \|u_0\|_{L^2_x} \lesssim \varepsilon^2 \quad . \tag{17}$$

On the other hand, since $\|w(0)\|_{H^1_\lambda} \lesssim \|u_0\|_{H^1_\lambda} + \|u_0\|^2_{H^1_\lambda} \lesssim \varepsilon^2$, by continuity we can assume that

$$\|u\|_{X^1_{T,\lambda}} \lesssim \varepsilon \qquad \text{and} \qquad \|w\|_{X^1_{T,\lambda}} \lesssim \varepsilon ,$$

on some small enough interval $[0,T] \subset [0,1]$. But (14) then gives $\|w\|_{X^1_{T,\lambda}} \lesssim \varepsilon^2$ and this last inequality together with (16)-(17) imply now that $\|u\|_{X^1_{T,\lambda}} \lesssim \varepsilon^2$. In a standard way this proves that T can be taken equal to 1 and thus we have

$$\|u\|_{X^1_{1,\lambda}} \lesssim \varepsilon^2 \quad .$$

Consider now $u_0 \in H^1_\lambda$ such that $\|u_0\|_{H^1_\lambda} \lesssim \varepsilon^2$. Approximating u_0 in H^1_λ by a sequence $\{u_{0,n}\} \subset H^\infty_\lambda$, it follows that the sequence of the emanating solutions $\{u_n\} \subset C(\mathbb{R}; H^\infty_\lambda)$ is bounded in $X^1_{1,\lambda}$. We can thus pass to the limit up to a subsequence and obtain the existence of a solution $u \in X^1_{1,\lambda}$ of (BO).

3.2.2 Continuity, uniqueness and regularity of the flow map

As we notice already in the introduction, one of the main differences with the problem on the real axis is that the gauge transformation is Lipschitz from the space of L^2 functions with zero mean value on the circle into L^∞. This property turns out to be crucial to get the uniqueness and the continuity of the flow. We first prove that the flow-map is Lipschitz on a small ball of H^1_λ. The continuity of $t \mapsto u(t)$ in H^1_λ will follow directly.

Let u_1 and u_2 be two solutions of (BO) in $X^1_{T,\lambda}$ associated with the initial data φ_1 and φ_2 in H^1_λ. We assume that they satisfy

$$\|u_i\|_{X^1_{T,\lambda}} \lesssim \varepsilon^2 , \quad i = 1, 2, \tag{18}$$

for some $0 < T \le 1$ and where ε is taken as above. We set

$$z = w_1 - w_2 = -iP_+(e^{-iF_1} u_1) + iP_+(e^{-iF_2} u_2)$$

with F_i is defined as F and where u is replaced by u_i. Obviously, z satisfies

$$\begin{aligned} z_t - iz_{xx} &= \partial_x P_+\Big[P_-(\partial_x u_1 - \partial_x u_2)W_1\Big] + \partial_x P_+\Big[P_-(\partial_x u_2)(W_1 - W_2)\Big] \\ &+ P_0(u_1^2)\, P_+\Big((u_1 - u_2)e^{-iF_1}\Big) + P_0(u_1^2)\, P_+\Big(u_2(W_1 - W_2)\Big) \\ &+ P_0\Big(z(u_1 + u_2)\Big)\, P_+(u_2)\, . \end{aligned} \tag{19}$$

Note that (18) clearly ensures that for $i = 1, 2$,

$$\|w_i\|_{X^1_{T,\lambda}} \lesssim \|u_i\|_{X^1_{T,\lambda}} (1 + \|u_i\|_{X^1_{T,\lambda}}) \lesssim \varepsilon^2\, . \tag{20}$$

As previously, it follows that

$$\|z\|_{X^1_{T,\lambda}} \lesssim \|z(0)\|_{H^1_\lambda} + \|A\|_{L^1_T H^1_\lambda} \tag{21}$$

where A denotes the right hand side of (19). Note first that,

$$\begin{aligned} \|z(0)\|_{H^1_\lambda} &\lesssim \|\varphi_1 - \varphi_2\|_{H^1_\lambda}\Big(1 + \|\varphi_1\|_{H^1_\lambda} + \|\varphi_2\|_{H^1_\lambda}\Big) \\ &+ \|e^{-iF_1(0)} - e^{-iF_2(0)}\|_{L^\infty} \|\varphi_2\|_{H^1_\lambda} (1 + \|\varphi_2\|_{H^1_\lambda}) \end{aligned} \tag{22}$$

with

$$\|e^{-iF_1(0)} - e^{-iF_2(0)}\|_{L^\infty_\lambda} \lesssim \|\partial_x^{-1}(\varphi_1 - \varphi_2)\|_{L^\infty_\lambda} \lesssim \lambda^{1/2}\, \|\varphi_1 - \varphi_2\|_{L^2_\lambda} \tag{23}$$

and so,

$$\|z(0)\|_{H^1_\lambda} \lesssim (1+\lambda^{1/2}\varepsilon^2)\|\varphi_1-\varphi_2\|_{H^1_\lambda}\,. \tag{24}$$

We give now an estimate for $\|A\|_{L^1_T H^1_\lambda}$. From Lemma 4.1 we easily obtain that

$$\begin{aligned}
\|\partial_x P_+\Big[P_-(\partial_x u_1-\partial_x u_2)W_1\Big]\|_{L^1_T H^1_\lambda} &\lesssim T^{1/2}\,\|u_1-u_2\|_{X^1_{T,\lambda}}\,\|w_1\|_{X^1_{T,\lambda}}\,,\\
\|\partial_x P_+\Big[P_-(\partial_x u_2)(W_1-W_2)\Big]\|_{L^1_T H^1_\lambda} &\lesssim T^{1/2}\,\|u_2\|_{X^1_{T,\lambda}}\,\|z\|_{X^1_{T,\lambda}}\,,\\
\|P_0(u_1^2)\,P_+\Big[(u_1-u_2)e^{-iF_1}\Big]\|_{L^1_T H^1_\lambda} &\lesssim T^{1/2}\,\|u_1-u_2\|_{X^1_{T,\lambda}}(1+\|u_1\|_{X^1_{T,\lambda}})\|u_1\|^2_{X^1_{T,\lambda}}\,,\\
\|P_0\Big(z(u_1+u_2)\Big)\,P_+(u_2)\|_{L^1_T H^1_\lambda} &\lesssim T^{1/2}\,\|z\|_{X^1_{T,\lambda}}\|u_2\|_{X^1_{T,\lambda}}(\|u_1\|_{X^1_{T,\lambda}}+\|u_2\|_{X^1_{T,\lambda}})\,,
\end{aligned}$$

and proceeding as for $\|z(0)\|_{H^1_\lambda}$,

$$\begin{aligned}
\|P_0(u_1^2)\,P_+(u_2(W_1-W_2))\|_{L^1_T H^1_\lambda} &\lesssim T^{1/2}\,\|u_1\|^2_{X^1_{T,\lambda}}\|u_2\|_{X^1_{T,\lambda}}\|W_1-W_2\|_{L^\infty_{T,x}}\\
&+\; T^{1/2}\,\|u_1\|^2_{X^1_{T,\lambda}}\|u_2\|_{X^1_{T,\lambda}}\|z\|_{X^1_{T,\lambda}}\\
&\lesssim T^{1/2}\lambda^{1/2}\,\|u_1\|^2_{X^1_{T,\lambda}}\|u_2\|_{X^1_{T,\lambda}}\|u_1-u_2\|_{L^\infty_T L^2_x}\\
&+\; T^{1/2}\,\|u_1\|^2_{X^1_{T,\lambda}}\|u_2\|_{X^1_{T,\lambda}}\|z\|_{X^1_{T,\lambda}}\,.
\end{aligned}\tag{25}$$

Hence gathering (20), (24) and the previous estimates we infer that,

$$\begin{aligned}
\|z\|_{X^1_{T,\lambda}} &\lesssim (1+\varepsilon^2\lambda^{1/2})\|\varphi_1-\varphi_2\|_{H^1_\lambda}\\
&+\; \varepsilon^2 T^{1/2}\Big(\|z\|_{X^1_{T,\lambda}}+(1+\lambda^2)\|u_1-u_2\|_{X^1_{T,\lambda}}\Big)\,.
\end{aligned}\tag{26}$$

On the other hand, we have

$$\begin{aligned}
u_1-u_2 &= \partial_x F_1-\partial_x F_2\\
&= ie^{iF_1}\Big[z+\partial_x P_-\Big(e^{-iF_1}-e^{-iF_2}\Big)\Big]+i(e^{iF_1}-e^{iF_2})\Big(w_2+\partial_x P_-(e^{-iF_2})\Big)
\end{aligned}$$

and thus

$$\begin{aligned}
P_+(u_1-u_2) &= iP_+(e^{iF_1}z)+iP_+\Big[e^{iF_1}\partial_x P_-\Big(e^{-iF_1}-e^{-iF_2}\Big)\Big]\\
&\quad +iP_+\Big[(e^{iF_1}-e^{iF_2})w_2\Big]+iP_+\Big[(e^{iF_1}-e^{iF_2})\partial_x P_-(e^{-iF_2})\Big]\quad .
\end{aligned}$$

Therefore, as in (16),

$$\begin{aligned}
\|u_1-u_2\|_{X^1_{T,\lambda}} &\lesssim \|z\|_{X^1_{T,\lambda}}(1+\|u_1\|_{X^1_{T,\lambda}})+\|u_1-u_2\|_{X^1_{T,\lambda}}\|u_2\|_{X^1_{T,\lambda}}\\
&\quad +\|e^{iF_1}-e^{iF_2}\|_{L^\infty_{T,\lambda}}\Big(\|u_1\|_{X^1_{T,\lambda}}+\|u_2\|_{X^1_{T,\lambda}}+\|u_1\|_{X^1_{T,\lambda}}\|u_2\|_{X^1_{T,\lambda}}\Big)\\
&\quad +\|w_2\|_{X^1_{T,\lambda}}\Big(\|e^{iF_1}-e^{iF_2}\|_{L^\infty_{T,\lambda}}(1+\|u_1\|_{X^1_{T,\lambda}})+\|u_1-u_2\|_{X^1_{T,\lambda}}\Big)\quad .
\end{aligned}$$

But, proceeding as in (23), we see that

$$\|e^{-iF_1}-e^{-iF_2}\|_{L^\infty_{T,\lambda}}\lesssim \lambda^{1/2}\|u_1-u_2\|_{L^\infty_T L^2_\lambda}\quad . \tag{27}$$

Writing now the equation satisfied by u_1-u_2 and using it's Duhamel formulation together with TT^* arguments and Sobolev inequalities we easily obtain that for $0<T\le 1$,

$$\|u_1-u_2\|_{L^\infty_T L^2_\lambda}\lesssim \|\varphi_1-\varphi_2\|_{L^2_\lambda}+T^{1/2}\|u_1-u_2\|_{X^1_{T,\lambda}}(\|u_1\|_{X^1_{T,\lambda}}+\|u_2\|_{X^1_{T,\lambda}})\quad . \tag{28}$$

Gathering these estimates and recalling (18), (20) we finally obtain

$$\begin{aligned}\|u_1-u_2\|_{X^1_{T,\lambda}} &\lesssim (1+\lambda^{1/2}\varepsilon^2)\|\varphi_1-\varphi_2\|_{H^1_\lambda}\\ &\quad +\varepsilon^2\|z\|_{X^1_{T,\lambda}}+\varepsilon^2\|u_1-u_2\|_{X^1_{T,\lambda}}+\lambda^{1/2}\varepsilon^2\|u_1-u_2\|_{L^\infty_T L^2_\lambda}\\ &\lesssim (1+\lambda^{1/2}\varepsilon^2)\|\varphi_1-\varphi_2\|_{H^1_\lambda}+\varepsilon^2\|u_1-u_2\|_{X^1_{T,\lambda}}\\ &\quad +T^{1/2}\,\varepsilon^4\lambda^{1/2}\|u_1-u_2\|_{X^1_{T,\lambda}}\quad .\end{aligned} \tag{29}$$

Hence, for $0<T\le T_\lambda\sim\lambda^{-1}$, we get

$$\|u_1-u_2\|_{X^1_{T,\lambda}}\lesssim (1+\varepsilon^2\lambda^{1/2})\|\varphi_1-\varphi_2\|_{H^1_\lambda}\quad . \tag{30}$$

With (30) in hand, we observe that the approximative sequence u^n constructed above is a Cauchy sequence in $C([0,T_\lambda];H^1_\lambda)$ since $\|u_n\|_{X^1_{1,\lambda}}\lesssim\varepsilon^2$ and so $u_{0,n}$ converges to u_0 in H^1_λ. Hence, u belongs to $C([0,T_\lambda];H^1_\lambda)$. Repeating this argument we get that actually $u\in C([0,1];H^1_\lambda)$. Moreover, (30) clearly ensures the uniqueness in the considered class and that the flow-map is Lipschitz from the ball of H^1_λ with radius ε^2 into $C([0,1];H^1_\lambda)$.

3.3 The case of arbitrary large initial data

Here we used the dilation symmetry of the equation to extend the result for arbitrary large data. First note that if $u(t,x)$ is a 2π-periodic solution of (BO) on $[0,T]$ with initial data u_0 then $u_\lambda(t,x)=\lambda^{-1}u(\lambda^{-2}t,\lambda^{-1}x)$ is a $2\pi\lambda$-periodic solution of (BO) on $[0,\lambda^2T]$ emanating from $u_{0,\lambda}=\lambda^{-1}\,u_0(\lambda^{-1}x)$.

Now, let $u_0\in H^1$. If $\|u_0\|_{H^1}\le\varepsilon^2$ we are in the small initial data case. Otherwise, we set

$$\lambda=\varepsilon^{-4}\|u_0\|^2_{H^1}\ge 1$$

so that $u_{0,\lambda}$ satisfies

$$\|u_{0,\lambda}\|_{H^1_\lambda}\lesssim\varepsilon^2\quad .$$

Hence we are reduce to the case of small initial data. Therefore, there exists a unique local solution $u_\lambda \in C([0,1];H^1_\lambda)\cap X^1_{1,\lambda}$ of (BO) emanating from $u_{0,\lambda}$. This proves the existence and uniqueness in $C([0,T];H^1)\cap X^1_{T,1}$ of the solution u emanating from u_0 with $T\sim \|u_0\|_{H^1}^{-4}$. The fact that the flow-map is Lipschitz on every bounded set of H^1 follows as well.

Finally, note that the change of unknown (11) preserves the continuity of the solution and the continuity of the flow-map in $H^1(\mathbb{T})$. Moreover, the Lipschitz property (on bounded sets) of the flow-map is also preserved on the hyperplans of $H^1(\mathbb{T})$ with fixed mean-value.

4 Gauge transform for (GBO) and nonlinear estimates

Let us now begin the proof of Theorem 2. As for the (BO) equation we have to perform a gauge transformation to obtain suitable estimates in $X^1_{T,\lambda}$ for regular solutions of (GBO).

4.1 The gauge transformation

For u a smooth $2\pi\lambda$-periodic solution of (GBO) we consider v defined as

$$v(t,x)=2^{1/k}\,u(t,x+\int_0^t ⨍ u^k)\,, \tag{31}$$

which satisfies

$$\begin{cases} \partial_t v+\mathcal{H}\partial_x^2 v = 2\,M(v^k)\partial_x v\ ,\ (t,x)\in\mathbb{R}\times\mathbb{T}\,,\\ v(0,x)=u_0(x)\ ,\end{cases}$$

where $M(g)=g-⨍ g$ (such a manipulation is used in [7] for the generalized Korteweg-de-Vries equations on the torus). In the same spirit as in [10] and [19], define w the gauge transform of u by

$$w=P_+(e^{-iF}v)\,,\ F(t,x)=\sum_{q\in\mathbb{Z}^*/\lambda} C_q(M(v^k))\frac{e^{iqx}}{iq}=\partial_x^{-1}(M(v^k))\ . \tag{32}$$

In the sequel our aim is to derive a suitable equation satisfied by w. Noticing that

$$\begin{cases} w_t = P_+[e^{-iF}(-iF_t v+v_t)]\,,\\ w_{xx}=P_+[e^{-iF}(-2iv_xF_x+v_{xx}-F_x^2v-iF_{xx}v)]\,,\end{cases}$$

we obtain that w solves the semilinear Schrödinger equation

$$w_t - iw_{xx} = P_+[e^{-iF}((v_t - iv_{xx} - 2F_x v_x) + (-F_{xx}v + iF_x^2 v) - iF_t v)] \,. \quad (33)$$

We compute now the three terms appearing in the right hand side of (33). First we have,

$$\begin{aligned} A &= v_t - iv_{xx} - 2F_x v \\ &= v_t + Hv_{xx} - 2iP_-(v_{xx}) - 2M(v^k)v_x \\ &= -2iP_-(v_{xx}) \,. \end{aligned} \quad (34)$$

Next we have,

$$\begin{aligned} B &= -F_{xx} + iF_x^2 v \\ &= -[M(v^k)]_x v + i[M(v^k)]^2 v \\ &= B_1 + B_2 \,. \end{aligned} \quad (35)$$

On the other hand $C = -ivF_t$ with,

$$\begin{aligned} F_t &= \sum_{q\in\mathbb{Z}^*/\lambda} C_q(M(v^k)_t)\frac{e^{iqx}}{iq} \\ &= k\sum_{q\in\mathbb{Z}^*/\lambda} C_q(v^{k-1}v_t)\frac{e^{iqx}}{iq} \\ &= -k\sum_{q\in\mathbb{Z}^*/\lambda} C_q(v^{k-1}Hv_{xx})\frac{e^{iqx}}{iq} + 2k\sum_{q\in\mathbb{Z}^*/\lambda} C_q(v^{k-1}M(v^k)v_x)\frac{e^{iqx}}{iq} \\ &= C_1 + C_2 \,, \end{aligned} \quad (36)$$

with,

$$\begin{aligned} C_1 &= -k\sum_{q\in\mathbb{Z}^*/\lambda} C_q(v^{k-1}Hv_{xx})\frac{e^{iqx}}{iq} \\ &= k\sum_{q\in\mathbb{Z}^*/\lambda}[\sum_{r,q-r\in\mathbb{Z}^*/\lambda}(q-r)^2 C_r(v^{k-1})C_{q-r}(Hv)]\frac{e^{iqx}}{iq} \\ &= -k\sum_{q\in\mathbb{Z}^*/\lambda}[\sum_{r,q-r\in\mathbb{Z}^*/\lambda} C_r(v^{k-1})C_{q-r}(Hv_x)]e^{iqx} \\ &+ k(k-1)\sum_{q\in\mathbb{Z}^*/\lambda}[\sum_{r,q-r\in\mathbb{Z}^*/\lambda} C_r(v^{k-2}v_x)C_{q-r}(Hv_x)]\frac{e^{iqx}}{iq} \end{aligned}$$

$$\begin{aligned}
&= -k \sum_{q\in\mathbb{Z}^*/\lambda} C_n(v^{k-1}Hv_x)e^{inx} + k(k-1)\sum_{q\in\mathbb{Z}^*/\lambda} C_q(v^{k-2}v_xHv_x)\frac{e^{iqx}}{iq} \\
&= -kM(v^{k-1}Hv_x) + k(k-1)\sum_{q\in\mathbb{Z}^*/\lambda} C_q(v^{k-2}v_xHv_x)\frac{e^{iqx}}{iq} \\
&= C_{1,1} + C_{1,2}\,. \qquad (37)
\end{aligned}$$

and,

$$\begin{aligned}
C_2 &= 2k\sum_{q\in\mathbb{Z}^*/\lambda} C_q(v^{k-1}M(v^k)v_x)\frac{e^{iqx}}{iq} \\
&= 2\sum_{q\in\mathbb{Z}^*/\lambda} C_q(M(v^k)M(v^k)_x)\frac{e^{iqx}}{iq} \\
&= \sum_{q\in\mathbb{Z}^*/\lambda} C_q((M(v^k)^2)_x)\frac{e^{iqx}}{iq} \\
&= M[M(v^k)^2]\,. \qquad (38)
\end{aligned}$$

Gathering (34)-(38)and noticing that

$$\begin{aligned}
B_1 - iv\,C_{1,2} &= M(v^k)_x v + ikvM(v^{k-1}Hv_x) \\
&= kvM(v^{k-1}P_+v_x) - kvM(v^{k-1}P_-v_x) - kv^kv_x \\
&= kvM(v^{k-1}P_+v_x) - kvM(v^{k-1}P_-v_x) - kvM(v^{k-1}v_x) \\
&= -2kvM(v^{k-1}v_x)\,, \qquad (39)
\end{aligned}$$

and that

$$\begin{aligned}
B_2 - iv\,C_2 &= -iv\,[M(M(v^k)^2) - M(v^k)^2] \\
&= -iv\,P_0(M(v^k)^2)\,, \qquad (40)
\end{aligned}$$

we infer that,

$$\begin{aligned}
w_t - iw_{xx} &= P_+(e^{-iF}v\,P_0(M(v^k)^2) \\
&- 2iP_+(e^{iF}P_-v_{xx}) \\
&- 2kP_+(e^{-iF}v\,M(v^{k-1}P_-v_x)) \\
&- k(k-1)P_+\Big(e^{-iF}v\sum_{q\in\mathbb{Z}^*/\lambda}\frac{C_q(v^{k-2}v_xHv_x)}{iq}\,e^{iqx}\Big) \\
&:= a+b+c+d\,. \qquad (41)
\end{aligned}$$

which gives us the Schrödinger equation satisfies by w.

4.2 Nonlinear estimates on the gauge transform

In this subsection we derived $X^1_{T,\lambda}$ and $X^2_{T,\lambda}$-estimates for w the gauge transform of v a $2\pi\lambda$-periodic solution of (GBO). Our aim is to prove the following estimate,

Proposition 4.1 *Let v a $X^1_{T,\lambda}$-solution of (GBO) and let us consider $w = P_+(e^{-iF}v)$. Then for $0 \le T \le 1$ and for $\lambda > 1$,*

$$\|w\|_{X^1_{T,\lambda}} \lesssim \|w_0\|_{H^1_\lambda} + T^{1/4}\,(\|v\|^{k+1}_{X^1_{T,\lambda}} + \|v\|^{2k+1}_{X^1_{T,\lambda}} + \|v\|^{3k+1}_{X^1_{T,\lambda}})\,. \tag{42}$$

Moreover there exists a polynomial function Q such that,

$$\|w\|_{X^2_{T,\lambda}} \lesssim \|w_0\|_{H^1_\lambda} + T^{1/4}\,Q(\|v\|_{X^1_{T,\lambda}})\,\|v\|_{X^2_{T,\lambda}}\,. \tag{43}$$

Proof. We will only prove (42) since (43) can be derived in exactly the same way (the linear dependence in the strong norm $\|u\|_{X^2_{T,\lambda}}$ of the right hand side of (43) is standard). Recall first that w solves

$$w_t - iw_{xx} = f = a + b + c + d\,,$$

where a, b, c and d are defined in (41). Hence from the periodic estimate

$$\|w\|_{L^\infty_t L^2_x} + \|w\|_{L^4_{t,x}} \le \|f\|_{L^1_t L^2_x} \tag{44}$$

we infer that

$$\|w\|_{X^1_{T,\lambda}} \;\lesssim\; \|a\|_{L^1_t L^2_x} + \|\partial_x a\|_{L^1_t L^2_x} + \ldots + \|d\|_{L^1_t L^2_x} + \|\partial_x d\|_{L^1_t L^2_x}\,. \tag{45}$$

Now we recall the following periodic version of of Lemma 3.4 in [18] which allows to share derivatives when estimating terms like $D^\alpha_x P_+[fP_-(D^\beta_x g)]$.

Lemma 4.1 *Let $\alpha > 0$, $\beta \ge 0$, $1 < p < +\infty$ and $1 \le q \le +\infty$. Then*

$$\|D^\alpha_x P_+[fP_-(D^\beta_x g]\|_{L^q_q L^p_x} \lesssim \|D^{\gamma_1}_x f\|_{L^{q_1}_t L^{p_1}_x} \|D^{\gamma_2} g\|_{L^{q_2}_t L^{p_2}_x}\,, \tag{46}$$

with $1 < p_i, q_i < +\infty$, $1/p_1 + 1/p_2 = 1/p$, $1/q_1 + 1/q_2 = 1/q$ and $\gamma_1 \ge \alpha$, $\gamma_1 + \gamma_2 = \alpha + \beta$.

. Estimates for $a = P_0[M(v^k)^2]\,P_+(e^{-iF}v)$. We have,

$$\begin{aligned}
\|a\|_{L^1_T L^2_x} &= |P_0[M(v^k)^2]|\,\|v\|_{L^1_T L^2_x} \\
&\lesssim T\,\|M(v^k)\|^2_{L^\infty_T L^2_x}\,\|v\|_{L^\infty_T L^2_x} \\
&\lesssim T\,\|v\|^{2k}_{L^\infty_T L^{2k}_x}\,\|v\|_{L^\infty_T L^2_x} \\
&\lesssim T\,\|v\|^{2k+1}_{X^1_{T,\lambda}}\,.
\end{aligned} \tag{47}$$

Next we have

$$
\begin{aligned}
\|\partial_x a\|_{L^1_t L^2_x} &\lesssim |P_0[M(v^k)^2]|\, \|\partial_x P_+(e^{-iF} v)\|_{L^1_t L^2_x} \\
&\lesssim \|v\|^{2k}_{X^1_{T,\lambda}} (\|M(v^k)v\|_{L^1_t L^2_x} + \|e^{-iF} v_x\|_{L^1_t L^2_x}) \\
&\lesssim \|v\|^{2k}_{X^1_{T,\lambda}} (\|M(v^k)\|_{L^\infty_{t,x}} \|v\|_{L^1_t L^2_x} + \|v_x\|_{L^1_t L^2_x}) \\
&\lesssim T\, \|v\|^{2k}_{X^1_{T,\lambda}} (\|v\|^{k+1}_{X^1_{T,\lambda}} + \|v\|_{X^1_{T,\lambda}})\,, \qquad (48)
\end{aligned}
$$

where we use that $\|M(v^k)\|_{L^\infty_{t,x}} \lesssim \|M(v^k)\|_{L^\infty_t H^1_x} \lesssim \|v\|^k_{L^\infty_t H^1_x}$.

. Estimates for $b = -2iP_+[e^{-iF} P_-(v_{xx})]$. ¿From Lemma 4.1 we have,

$$
\begin{aligned}
\|b\|_{L^1_t L^2_x} &\lesssim \|\partial_x e^{-iF}\|_{L^\infty_t L^4_x} \|v_x\|_{L^1_t L^4_x} \\
&\lesssim T^{3/4} \|M(v^k)\|_{L^\infty_t L^4_x} \|v_x\|_{L^4_{t,x}} \\
&\lesssim T^{3/4} \|M(v^k)\|_{L^\infty_t H^1_x} \|v_x\|_{L^4_{t,x}} \\
&\lesssim T^{3/4} \|v\|^{k+1}_{X^1_{T,\lambda}}\,. \qquad (49)
\end{aligned}
$$

Now, again from Lemma 4.1, we have

$$
\begin{aligned}
\|\partial_x b\|_{L^1_t L^2_x} &\lesssim \|\partial_{xx}(e^{-iF})\|_{L^4_{t,x}} \|v_x\|_{L^{4/3}_t L^4_x} \\
&\lesssim T^{2/3} \|v^{k-1} v_x\|_{L^4_{t,x}} \|v_x\|_{L^4_{t,x}} \\
&\lesssim T^{2/3} \|v^{k-1}\|_{L^\infty_{t,x}} \|v_x\|^2_{L^4_{t,x}} \\
&\lesssim T^{2/3} \|v\|^{k-1}_{L^\infty_t H^1_x} \|v_x\|_{L^4_{t,x}} \\
&\lesssim T^{2/3} \|v\|^{k+1}_{X^1_{T,\lambda}}\,. \qquad (50)
\end{aligned}
$$

. Estimates for $c = -2kP_+(e^{-iF} v\, M[v^{k-1} P_-(v_x)])$.

$$
\begin{aligned}
\|c\|_{L^1_t L^2_x} &\lesssim \|v\, M[v^{k-1} P_-(v_x)]\|_{L^1_t L^2_x} \\
&\lesssim \|v\|_{L^\infty_{t,x}} \|v^{k-1} P_-(v_x)\|_{L^1_t L^2_x} \\
&\lesssim \|v\|_{L^\infty_t H^1_x} \|v^{k-1}\|_{L^\infty_{t,x}} \|v_x\|_{L^1_t L^2_x} \\
&\lesssim T\, \|v\|^{k+1}_{L^\infty_t H^1_x} \\
&\lesssim T\, \|v\|^{k+1}_{X^1_{T,\lambda}}\,. \qquad (51)
\end{aligned}
$$

Next from obvious calculation, the Sobolev embeding (in space) $H^1 \hookrightarrow L^\infty$ and Lemma 4.1 we infer that

$$
\|\partial_x c\|_{L^1_t L^2_x} \lesssim \|M(v^k)\, v\, M[v^{k-1} P_-(v_x)]\|_{L^1_t L^2_x} + \|v_x M[v^{k-1} P_-(v_x)]\|_{L^1_t L^2_x}
$$

$$
\begin{aligned}
&+ \quad \|v^{k-1}\, v_x P_-(v_x)\|_{L^1_t L^2_x} + T^{1/2}\,\|P_+[e^{-iF}\, v^k P_-(v_{xx})]\|_{L^2_t L^2_x} \\
&\lesssim \quad \|v\|^{2k}_{X^1_{T,\lambda}}\,\|v_x\|_{L^1_t L^2_x} + \|v\|^{k-1}_{X^1_{T,\lambda}}\|v_x\|^2_{L^2_t L^4_x} \\
&+ \quad T^{1/2}\,\|\partial_x(e^{-iF} v^k)\|_{L^4_{t,x}}\,\|P_-(v_x)\|_{L^4_{t,x}} \\
&\lesssim \quad T\,\|v\|^{2k+1}_{X^1_{T,\lambda}} + T^{1/2}\|v\|^{k+1}_{X^1_{T,\lambda}}\,(1+\|v\|^k_{X^1_{T,\lambda}})\,. \qquad (52)
\end{aligned}
$$

. Estimates for $d = -k(k-1)P_+[vhe^{-iF}]$ where

$$
h = \sum_{q\in\mathbb{Z}^*/\lambda} \frac{C_q(v^{k-2}v_x H v_x)}{iq}\, e^{iqx}\,. \qquad (53)
$$

We note first that,

$$
\begin{aligned}
\|d\|_{L^1_t L^2_x} &\lesssim \|v\|_{L^1_t L^2_x}\,\|h\|_{L^\infty_{t,x}} \\
&\lesssim T\,\|v\|_{L^\infty_t L^2_x}\,\|v^{k-2}v_x H v_x\|_{L^\infty_t L^1_x} \\
&\lesssim T\,\|v\|_{X^1_{T,\lambda}}\,\|v^{k-1}\|_{L^\infty_{t,x}}\|v_x\|^2_{L^\infty_t L^2_x} \\
&\lesssim T\,\|v\|^{k+1}_{X^1_{T,\lambda}}\,.
\end{aligned} \qquad (54)
$$

On the other hand, in the same way than previously, we see that

$$
\begin{aligned}
\|\partial_x d\|_{L^1_t L^2_x} &\lesssim \|v_x h\|_{L^1_t L^2_x} + \|M(v^k) v h\|_{L^1_t L^2_x} + \|v \sum_{q\in\mathbb{Z}^*/\lambda} C_q(v^{k-2}v_x H v_x) e^{iqx}\|_{L^1_t L^2_x} \\
&\lesssim \|v_x\|_{L^1_t L^2_x}\,\|v^{k-2}v_x H v_x\|_{L^\infty_t L^1_x} \\
&+ \|M(v^k)\|_{L^\infty_{t,x}}\|v\|_{L^1_t L^2_x}\|v^{k-2}v_x H v_x\|_{L^\infty_t L^1_x} \\
&+ T^{1/2}\|v \sum_{q\in\mathbb{Z}^*/\lambda} C_q(v^{k-2}v_x H v_x) e^{iqx}\|_{L^2_{t,x}} \\
&\lesssim T\,\|v\|^{k+1}_{X^1_{T,\lambda}} + T\,\|v\|^{2k+1}_{X^1_{T,\lambda}} + T^{1/2}\|v\|_{L^\infty_{t,x}}\,\|v^{k-2}\|_{L^\infty_{t,x}}\|v_x H v_x\|_{L^2_{t,x}} \\
&\lesssim T\,\|v\|^{k+1}_{X^1_{T,\lambda}} + T\,\|v\|^{2k+1}_{X^1_{T,\lambda}} + T^{1/2}\|v\|^{k-1}_{L^\infty_{t,x}}\,\|v_x\|^2_{L^4_{t,x}} \\
&\lesssim (T+T^{1/2})\,\|v\|^{k+1}_{X^1_{T,\lambda}} + T\,\|v\|^{2k+1}_{X^1_{T,\lambda}}\,.
\end{aligned} \qquad (55)
$$

5 Local well-posedness for (GBO)

5.1 Local well-posedness for "small" initial data

5.1.1 Nonlinear Estimates on u

As for the (BO) equation, we first state the result for "small" data. More precisely, for any real number $A > 1$ given, we first prove the local well-posedness result for $u_0 \in H^1_\lambda$ such that

$$\|u_0\|_{L^2_\lambda} \lesssim A\,\varepsilon^{1/k-1/2} \quad \text{and} \quad \|\partial_x u_0\|_{L^2_\lambda} \lesssim \varepsilon^{1/2+1/k} \tag{56}$$

for some $0 < \varepsilon = \varepsilon(A) < 1$ which does not depend on λ. Note that the L^2-norm of u_0 may be taken arbitrary large in (56). We stress out the attention of the reader that this will be necessary to consider such initial data since the L^2-norm is subcritical for (GBO) as soon as $k \geq 2$ and since we will use some dilation arguments in the case of non small initial data.

So let us consider $u_0 \in H^\infty_\lambda$ satisfying (56) and let u be the emanating maximal solution given for instance by [1]. We are going to prove that $u \in C([0,1]; H^\infty_\lambda)$ with

$$\|u\|_{X^0_{1,\lambda}} \lesssim A\,\varepsilon^{1/k-1/2} \quad \text{and} \quad \|\partial_x u\|_{X^0_{1,\lambda}} \lesssim \varepsilon^{1/2+1/k} \quad . \tag{57}$$

Notice that despite the local existence is only known for H^2_λ initial data, it will be enough to derive $X^1_{T,\lambda}$ estimates for u since we will check at the end of this subsection that for $0 < T \leq 1$, $\|u\|_{L^\infty_T H^2_\lambda}$-norm cannot blow up as long as $\|u\|_{X^1_{T,\lambda}}$ remains bounded. Therefore, according to the local well-posedness result in [1], the solution can be extended in $C([0,T]; H^\infty_T)$ as soon as $\|u\|_{X^1_{T,\lambda}} < \infty$.

First, by a continuity argument, we can assume that $\|\partial_x u\|_{X^0_{T,\lambda}} \lesssim \varepsilon^{1/2}$ for some $0 < T < T^*$ where T^* is the maximal time of existence of the solution u. Next, since the L^2-norm of u is a constant of the motion, to prove the desired result, it suffices to prove that if

$$\|u\|_{L^\infty_T L^2_\lambda} \lesssim A\,\varepsilon^{1/k-1/2} \quad \text{and} \quad \|\partial_x u\|_{X^0_{T,\lambda}} \lesssim \varepsilon^{1/2}, \tag{58}$$

with $0 < T < 1$, then

$$\|u\|_{L^4_{T,\lambda}} \lesssim A\,\varepsilon^{1/k-1/2} \quad \text{and} \quad \|\partial_x u\|_{X^0_{T,\lambda}} \lesssim \varepsilon^{1/2+1/k} \quad . \tag{59}$$

The estimate on $\|u\|_{L^4_{1,\lambda}}$ is trivially satisfied since by Sobolev inequalities and interpolation,

$$\|u\|_{L^4_{1,\lambda}} \lesssim \|u\|_{L^\infty_1 \dot{H}^{1/4}_\lambda} \lesssim \|u\|^{3/4}_{L^\infty_1 L^2_\lambda} \|u_x\|^{1/4}_{L^\infty_1 L^2_\lambda} \lesssim \varepsilon^{3/(4k)-1/4} \lesssim \varepsilon^{1/k-1/2} \quad .$$

We now estimate the $L^\infty_T L^2_\lambda$-norm of u_x. Recall that since u is real valued,

$$\|u_x\|_{L^\infty_T L^2_\lambda} \le \|P_1(u_x)\|_{L^\infty_T L^2_\lambda} + 2\,\|P_+(u_x)\|_{L^\infty_T L^2_\lambda} \tag{60}$$

with

$$P_+(u_x) = P_+(\partial_x(e^{iF}w)) + P_+\Big(\partial_x[e^{iF}P_-(e^{-iF}u)]\Big)\,, \tag{61}$$

We consider the first term in the right hand side of (60). Since u solves (GBO), it follows from Bernstein inequalities that,

$$\begin{aligned}
\|P_1 u_x\|_{L^\infty_T L^2_\lambda} &\lesssim \|V(t)\partial_x u_0\|_{L^2_\lambda} + \Big\|\int_0^t V(t-t')u^{k+1}(t')\,dt'\Big\|_{L^\infty_T L^2_\lambda}\\
&\lesssim \|\partial_x u_0\|_{L^2_\lambda} + T\,\|u\|^{k+1}_{L^\infty_T L^{2(k+1)}_\lambda}\\
&\lesssim \|\partial_x u_0\|_{L^2_\lambda} + T\,\|u\|^{k+1}_{X^1_{T,\lambda}}\\
&\lesssim \varepsilon^{1/k+1/2}
\end{aligned} \tag{62}$$

choosing $T = T(\varepsilon) > 0$ small enough.

We consider now the second term in the right hand side of (60) by means of the decomposition given by (61). From (42) in Proposition 4.1 we infer that for $T = T(\varepsilon) > 0$ small enough,

$$\begin{aligned}
\|\partial_x P_+(e^{iF}w)\|_{L^\infty_T L^2_\lambda} &\lesssim \|F_x\, w\|_{L^\infty_T L^2_\lambda} + \|\partial_x w\|_{L^\infty_T L^2_\lambda}\\
&\lesssim \|u\|^{k+1}_{L^\infty_T L^{2(k+1)}_\lambda} + \|\partial_x w_0\|_{L^2_\lambda} + T^{1/4}\,\|u\|^{3k+1}_{X^1_{T,\lambda}}\\
&\lesssim \|u\|^{k+1}_{L^\infty_T L^{2(k+1)}_\lambda} + \|u_0\|^{k+1}_{L^{2(k+1)}_\lambda} + \|\partial_x u_0\|_{L^2_\lambda} + \varepsilon^{1/2+1/k}\\
&\lesssim \|u\|^{k+1}_{L^\infty_T L^{2(k+1)}_\lambda} + \|u_0\|^{k+1}_{L^{2(k+1)}_\lambda} + \varepsilon^{1/2+1/k}\,.
\end{aligned} \tag{63}$$

Moreover, using Lemma 4.1 we see that

$$\begin{aligned}
\Big\|\partial_x P_+\Big(e^{iF}\,P_-(e^{-iF}u)\Big)\Big\|_{L^\infty_T L^2_\lambda} &\lesssim \|\partial_x F\|_{L^\infty_T L^{2(k+1)/k}_\lambda}\,\|u\|_{L^\infty_T L^{2(k+1)}_\lambda}\\
&\lesssim \|u\|^{k+1}_{L^\infty_T L^{2(k+1)}_\lambda}\quad.
\end{aligned} \tag{64}$$

It thus remains to get a good estimate on $\|u_0\|^{k+1}_{L^{2(k+1)}_\lambda}$ and $\|u\|^{k+1}_{L^\infty_T L^{2(k+1)}_\lambda}$. To do this we first note that we have,

$$\begin{aligned}
\|u_0\|^{k+1}_{L^{2(k+1)}_\lambda} &\lesssim \|u_0\|^{k+1}_{\dot H^{k/[2(k+1)]}_\lambda}\\
&\lesssim \|u_0\|^{k/2}_{\dot H^1_\lambda}\,\|u_0\|^{(k+2)/2}_{L^2_\lambda}\\
&\lesssim \varepsilon^{(1/2+1/k)k/2}\,\varepsilon^{(1/k-1/2)(k+2)/2}\\
&\lesssim \varepsilon^{1/2+1/k}\,.
\end{aligned} \tag{65}$$

Concerning $\|u\|^{k+1}_{L^\infty_T L^{2(k+1)}_\lambda}$, remark that,

$$
\begin{aligned}
\|u\|^{k+1}_{L^\infty_T L^{2(k+1)}_\lambda} &\lesssim \|P_1 u\|^{k+1}_{L^\infty_T L^{2(k+1)}_\lambda} + \|\partial_x P_{>1} u\|^{k+1}_{L^\infty_T L^2_\lambda} \\
&\lesssim \|P_1 u\|^{k+1}_{L^\infty_T L^{2(k+1)}_\lambda} + \varepsilon^{(k+1)/2} \ . \qquad (66)
\end{aligned}
$$

But using that u is a solution of (GBO), Sobolev, Bernstein and Strichartz estimates, (56) and (58), we infer that

$$
\begin{aligned}
\|P_1 u\|^{k+1}_{L^\infty_T L^{2(k+1)}_\lambda} &\lesssim \|V(t)u_0\|^{k+1}_{\dot H^{k/[2(k+1)]}_\lambda} + \Big\| \int_0^t V(t-t')u^{k+1}(t')\,dt' \Big\|^{k+1}_{L^\infty_T L^2_\lambda} \\
&\lesssim \|u_0\|^{k+1}_{\dot H^{k/[2(k+1)]}_\lambda} + T^{k+1}\,\|u\|^{(k+1)}_{L^\infty_T L^{2(k+1)}_\lambda} \\
&\lesssim \varepsilon^{1/k+1/2} + T^{k+1}\,\|u\|^{(k+1)}_{L^\infty_T L^{2(k+1)}_\lambda} \qquad (67)
\end{aligned}
$$

since the operators $\partial_x P_1$ and $V(\cdot)$ are respectively bounded from $L^{2(k+1)}$ to L^2 and from L^2 to itself. This ensures that for $T = T(\varepsilon) > 0$ small enough,

$$
\|P_1 u\|^{k+1}_{L^\infty_T L^{2(k+1)}_\lambda} \lesssim \varepsilon^{1/k+1/2} \quad . \qquad (68)
$$

Gathering (60)-(70), we thus obtain for $0 < T < 1$,

$$
\|\partial_x u\|_{L^\infty_T L^2_\lambda} \lesssim \varepsilon^{1/2+1/k} \ . \qquad (69)
$$

Let us now estimate the $L^4_{T,\lambda}$-norm of u_x. Proceeding in the same way than previously we first infer that

$$
\begin{aligned}
\|P_1 u_x\|_{L^4_{T,\lambda}} &\lesssim \|V(t)\partial_x u_0\|_{L^4_{T,\lambda}} + \Big\| \int_0^t V(t-t')u^{k+1}(t')\,dt' \Big\|_{L^4_{T,\lambda}} \\
&\lesssim \|\partial_x u_0\|_{L^2_\lambda} + T\,\|u\|_{L^\infty_T L^{4(k+1)}_\lambda} \\
&\lesssim \varepsilon^{1/k+1/2} \qquad (70)
\end{aligned}
$$

choosing $T = T(\varepsilon) > 0$ small enough.
Next we have,

$$
\begin{aligned}
\|\partial_x P_+(e^{iF}w)\|_{L^4_{T,\lambda}} &\lesssim \|F_x\, w\|_{L^4_{T,\lambda}} + \|\partial_x w\|_{L^4_{T,\lambda}} \\
&\lesssim \|u\|^{k+1}_{L^{4(k+1)}_{T,\lambda}} + \|u_0\|^{k+1}_{L^{2(k+1)}_\lambda} \\
&+ \|\partial_x u_0\|_{L^2_\lambda} + T^{1/4}\|u\|^{k+1}_{X^1_{T,\lambda}} \quad . \qquad (71)
\end{aligned}
$$

This combines with (69) and (61) completes the proof of (57), since by (56), (69) and Sobolev inequalities,

$$\begin{aligned}
\|u\|^{k+1}_{L^{4(k+1)}_{T,\lambda}} &\lesssim \|u\|^{k+1}_{L^\infty_T \dot H_\lambda^{(2k+1)/[4(k+1)]}} \\
&\lesssim \|u\|^{(2k+1)/4}_{L^\infty_T \dot H^1_\lambda} \|u\|^{(2k+3)/4}_{L^\infty_T L^2_\lambda} \\
&\lesssim \varepsilon^{(1/2+1/k)(2k+1)/4}\, \varepsilon^{(1/k-1/2)(2k+3)/4} \\
&\lesssim \varepsilon^{3/4+1/k} \ . \qquad (72)
\end{aligned}$$

It remains to check that $\|u_\lambda\|_{X^2_{T,\lambda}}$ can not go to infinity as long as $\|u_\lambda\|_{X^1_{T,\lambda}}$ remains bounded. First we have (here we use (43)),

$$\begin{aligned}
\|\partial_x^2 P_+(e^{-iF}w)\|_{X^0_{T,\lambda}} &\lesssim \|\partial_x F\, \partial_x w\|_{X^0_{T,\lambda}} + \|\partial_x^2 w\|_{X^0_{T,\lambda}} \\
&+ \|\partial_x^2 F\, w\|_{X^0_{T,\lambda}} + \|(\partial_x F)^2\, w\|_{X^0_{T,\lambda}} \\
&\lesssim \|\partial_x^2 w\|_{X^0_{T,\lambda}} + \|u\|^k_{L^\infty_{T,\lambda}} (1+\|u\|^k_{L^\infty_{T,\lambda}})\|w\|_{X^1_T} \\
&+ \|u\|^{k-1}_{L^\infty_{T,\lambda}} \|w\|_{L^\infty_{T,\lambda}} \|\partial_x u\|_{X^0_{T,\lambda}} \\
&\lesssim \|u\|^k_{X^1_{T,\lambda}} (1+\|u_\lambda\|^k_{X^1_{T,\lambda}})\|w\|_{X^1_{T,\lambda}} \\
&+ \|\partial_x^2 w_0\|_{L^2_\lambda} + T^{1/4}\, Q(\|u\|_{X^1_{T,\lambda}})\|u\|_{X^2_{T,\lambda}} \ . \qquad (73)
\end{aligned}$$

Furthermore,

$$\begin{aligned}
\Big\|\partial_x^2 P_+\Big(e^{iF}\, P_-(e^{-iF}u)\Big)\Big\|_{X^0_{T,\lambda}} &\lesssim (\|\partial_x^2 F\|_{L^\infty_T L^2_\lambda} + \|\partial_x F\|^2_{L^\infty_T L^4_\lambda})\|u\|_{L^\infty_{T,\lambda}} \\
&\lesssim \|u\|^{k+1}_{X^1_{T,\lambda}} (1+\|u\|^k_{X^1_{T,\lambda}}) \quad . \qquad (74)
\end{aligned}$$

Hence, gathering (73)-(74), we infer that

$$\|u\|_{X^2_{T,\lambda}} \lesssim \|u_0\|_{H^2}(1+\|u_0\|^{2k}_{H^2}) + \|u\|_{X^1_{T,\lambda}}(1+\|u\|^{2k+1}_{X^1_{T,\lambda}}) + T^{1/4} Q(\|u\|_{X^1_{T,\lambda}})\, \|u\|_{X^2_{T,\lambda}} \ . \qquad (75)$$

This completes the proof of (57).

5.1.2 Local existence

Now, let $u_0 \in H^1_\lambda$ satisfying (56) and let $\{u_0^n\} \subset H^\infty_\lambda$ a sequence converging to u_0 in H^1_λ. We denote by u_n the solution of (GBO) emanating from u_0^n. From the previous results $u_n \in C([0,1]; H^\infty_\lambda)$ and moreover, $\|u_n\|_{X^1_{1,\lambda}} \lesssim \varepsilon^{1/k-1/2}$ uniformly in n. Thus we can pass to the limit up to a subsequence which leads to the existence of a solution $u \in X^1_{1,\lambda}$ of (GBO) with u_0 as initial data.

5.1.3 Continuity, uniqueness and regularity of the flow map

As for the (BO) equation we first prove that the flow-map is Lipschitz on a small ball of H^1_λ. The continuity of $t \mapsto u(t)$ in H^1_λ will follow directly.

Let u_1 and u_2 be two solutions of (GBO) in $X^1_{T,\lambda}$, associated with the initial data φ_1 and φ_2 in H^1_λ. We assume that they satisfy

$$\|u_i\|_{X^1_{T,\lambda}} \lesssim A\,\varepsilon^{1/k-1/2} \quad \text{and} \quad \|\partial_x u_i\|_{X^0_{T,\lambda}} \lesssim \varepsilon^{1/2+1/k}\ , i=1,2 \quad , \tag{76}$$

where $0 < \varepsilon = \varepsilon(A) < 1$ has the same value as in (56). We then consider

$$z = w_1 - w_2 = -iP_+(e^{-iF_1}u_1) + iP_+(e^{-iF_2}u_2)$$

with F_i defined as F with u replaced by u_i. Following the calculations performed in Subsection 4.2 we clearly have,

$$\|z\|_{X^1_{T,\lambda}} \lesssim \|\varphi_1-\varphi_2\|_{H^1}(1+\|\varphi_1\|^k_{H^1}) + T^\nu\, \|z\|_{X^1_{T,\lambda}}(\|u_1\|^k_{X^1_{T,\lambda}} + \|u_2\|^k_{X^1_{T,\lambda}}) \tag{77}$$

Setting

$$\Lambda = \sum_{i=1}^{2} \|u_i\|_{L^\infty_T L^{2(k+1)}_\lambda} + \|u_i\|_{L^{4(k+1)}_{T,\lambda}} \quad ,$$

clearly, $\|w_i\|_{L^\infty_T L^{2(k+1)}_\lambda} + \|w_i\|_{L^{4(k+1)}_{T,\lambda}} \lesssim \Lambda$, and by (57) and Sobolev inequalities,

$$\Lambda \lesssim \varepsilon^{1/k - \frac{1}{2(k+1)}} \quad . \tag{78}$$

From (61) and Lemmas 4.1, we get after straightforward computations,

$$\begin{aligned} \|\partial_x(u_1-u_2)\|_{X^0_{T,\lambda}} \quad &\lesssim \quad \|\partial_x z\|_{X^0_\lambda}(1+\Lambda^k) + \Lambda^k \|u_1-u_2\|_{X^1_{T,\lambda}} \\ &\quad + \|(e^{iF_1} - e^{iF_2})\|_{L^\infty_{T,\lambda}} \Big(\|\partial_x w_2\|_{X^0_{T,\lambda}} + \Lambda^{k+1} \Big)\ . \end{aligned} \tag{79}$$

On the other hand,

$$\|F_1 - F_2\|\|_{L^\infty_{T,\lambda}} \lesssim \|u_1^k - u_2^k\|_{L^\infty_T L^1_\lambda} \lesssim \|u_1-u_2\|_{L^\infty_T L^k_\lambda}(\|u_1\|_{L^\infty_T L^k_\lambda} + \|u_2\|_{L^\infty_T L^k_\lambda})^{k-1}\ .$$

Hence, noticing that by Sobolev inequality,

$$\|u_i\|_{L^\infty_T L^k_\lambda} \lesssim \|u_i\|^{1/2+1/k}_{L^\infty_T L^2_\lambda} \|\partial_x u_i\|^{1/2-1/k}_{L^\infty_T L^2_\lambda} \lesssim 1, \quad i=1,2 \quad ,$$

we thus obtain that,

$$\|e^{iF_1} - e^{iF_2}\|_{L^\infty_{T,\lambda}} \lesssim \|F_1 - F_2\|_{L^\infty_{T,\lambda}} \lesssim \|u_1 - u_2\|_{X^1_{T,\lambda}}\ . \tag{80}$$

Noticing also that

$$\|\partial_x w_2\|_{X^0_{T,\lambda}} \lesssim \|\partial_x u_2\|_{X^0_{T,\lambda}} + \Lambda^{k+1} \lesssim \varepsilon^{1/2+1/k} \ ,$$

and

$$\|(u_1-u_2)\|_{X^0_{T,\lambda}} \lesssim \|u_1-u_2\|_{L^2_\lambda}+T^{1/4}\|u_1-u_2\|_{X^1_{T,\lambda}}(\|u_1\|_{X^1_{T,\lambda}}+\|u_2\|_{X^1_{T,\lambda}}) \quad (81)$$

we finally deduce from (77)-(81) that

$$\|u_1 - u_2\|_{X^1_{T,\lambda}} \lesssim \|\varphi_1 - \varphi_2\|_{H^1_\lambda}(1 + \|\varphi_1\|_{H^1_\lambda}) \quad . \qquad (82)$$

Combining (82) with the same arguments as in the end of Section 3.2, we obtain that the solution u constructed above belongs to $X^1_{1,\lambda} \cap C([0,1], H^1_\lambda)$ and is unique in this class. Moreover, the flow-map is Lipschitz from the ball of H^1_λ

$$\{\varphi \in H^1_\lambda, \quad \|\varphi\|_{L^2_\lambda} \le A\varepsilon^{1/k-1/2}, \ \|\varphi_x\|_{L^2_\lambda} \le \varepsilon^{1/2+1/k}\}, \ \text{ with } \varepsilon = \varepsilon(A),$$

into $X^1_{1,\lambda} \cap C([0,1], H^1_\lambda)$

5.2 Arbitrary large initial data

We used the dilation symmetry argument to extend the result for arbitrary large data. First note that if $u(t,x)$ is a 2π-periodic solution of (GBO) on $[0,T]$ with initial data u_0, then $u_\lambda(t,x) = \lambda^{-1/k}u(\lambda^{-2}t, \lambda^{-1}x)$ is a $2\pi\lambda$-periodic solution of (GBO) on $[0, \lambda^2 T]$ emanating from the initial data $u_{0,\lambda} = \lambda^{-1/k}u_0(\lambda^{-1}x)$.

Let $u_0 \in H^1$. If $\|\partial_x u_0\|_{L^2} \le \varepsilon^{1/2+1/k}$ with

$$\varepsilon = \varepsilon[(\|\partial_x u_0\|_{L^2}^{\frac{k-2}{k+2}} + 1)\|u_0\|_{L^2}] < 1 \ ,$$

then u_0 satisfies (57) with $A = (\|\partial_x u_0\|_{L^2}^{\frac{k-2}{k+2}} + 1)\|u_0\|_{L^2}$ and so we are done. Otherwise, we set

$$\lambda = \varepsilon^{-1}\|\partial_x u_0\|_{L^2}^{\frac{2k}{k+2}} \ge 1 \ ,$$

so that $u_{0,\lambda}$ satisfies (57) with ε and A defined as above. We are thus reduced to the case of small initial data. Therefore, there exists a unique local solution $u_\lambda \in C([0,1], H^1_\lambda) \cap X^1_{1,\lambda}$ of (GBO) emanating from $u_{0,\lambda}$. This proves the existence and uniqueness in $C([0,T], H^1) \cap X^1_T$ of the solution u emanating from u_0 with $T = T(\|u_0\|_{H^1})$ and $T \to +\infty$ as $\|u_0\|_{H^1} \to 0$. The

fact that the flow-map is Lipschitz on every bounded set of H^1 follows as well.

Finally, note that the change of unknown (31) preserves the continuity of the solution and the continuity of the flow-map in H^1. Moreover, for $k=2$, the flow-map is Lipschitz on every closed set S_β of $H^1(\mathbb{T})$ of the form

$$S_\beta = \{\varphi \in H^1(\mathbb{T}), \quad ⨍ \varphi^2 = \beta \quad \} \quad .$$

On the other hand, it does not preserve the Lipschitz property of the flow. Therefore, contrary to the real line case (cf.[19]), we do not know if the flow-map is Lipschitz or even uniformly continuous on bounded set. Recall that on the real line, the flow-map is known to be real-analytic on a small ball of $H^1(\mathbb{R})$ (cf. [15] and [18]) and Lipschitz on every bounded set of $H^1(\mathbb{R})$ (cf. [19]).

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